2-D Envelope Finite Element (EVFE) Technique- Implementation of PML and dispersive medium

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Abstract —**An implementation of perfect matched layer (PML) boundary conditions for 2-D envelope finite element (EVFE) techniques is proposed. The performance of PML is tested for different excitation modes and number of layers. It shows overall 50dB absorption with a 16 layers absorber. Furthermore, the extended EVFE formulation for structures containing dispersive medium is also presented. Numerical results generated show good agreements with analytical results.**

I. INTRODUCTION

It has been well-known that time-domain techniques have higher computational efficiency for broadband simulation while frequency domain approaches are more efficient for narrow band cases. Recently a novel technique called envelope finite element (EVFE) is proposed to simulate the time-domain envelopes of electromagnetic waves. It has been demonstrated that EVFE technique is computational efficient for both narrow band and broadband problems. Compared to time domain techniques, it can use much sparser time steps that are only limited by the bandwidth of the signal. Compared to frequency domain techniques, it requires only one time generation and inversion of finite element matrix over the interested frequency band. It is expected that this technique can be widely used for modeling of modern wireless and optical communication components where the modulation signal bandwidth is only a fraction of the carrier frequency.

In the original formulation proposed in [1], the absorbing boundary condition (ABC) is developed for termination of computational area. This limits the computational efficiency and applicability of EVFE techniques to complicated EM structures. As in other partial differential equation based numerical techniques, the ideal choice for the mesh truncation would be perfectly matched layer (PML) [2][3]. In this paper, the PML formulation for EVFE technique is developed and an overall performance of 50dB absorption is achieved for a broad range of incident angles. Furthermore, the formulation is extended to include the structures with dispersive medium [4][5]. To model the memory effects of the dispersive medium, instead of recursive convolution employed in other time-domain techniques, a mutual differential method is proposed and is verified to be computational efficient and accurate for EVFE techniques.

This paper is organized as follows. Section II presents the EVFE formulation implementing the anisotropic PML. Two 2-D numerical examples are presented here to testify the validity of the formulations. In section III, we use EVFE with PML to analyze plasma, a simple dispersive medium. Finally, conclusions are drawn in Section IV.

II. PML FOR EVFE FORMULATION

A general time-harmonic form of Maxwell equations in PML regions can be written as:

$$\begin{aligned} \nabla\times\vec{H} &= j\omega\varepsilon_0\varepsilon_r\overline{\overline{\varepsilon}}\vec{E}+\vec{J}_i \\ \nabla\times\vec{E} &= -j\omega\mu_0\mu_r\overline{\overline{\mu}}\vec{H} \end{aligned} \quad (1)$$

where

$$\overline{\overline{\varepsilon}} = \overline{\overline{\mu}} = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \quad (2)$$

and $$s_i = 1+\frac{\sigma_i}{j\omega\varepsilon_0}, \qquad i=x,y,z \quad (3)$$

For simplicity, we just consider the 2-D TM or TEM case with PML set in one direction, say, x direction, therefore, (1) is reduced to:

$$\frac{1}{\varepsilon_r}\nabla^2 H_z+\frac{\mu_r\omega^2}{c_0^2}s_x^2 H_z+\frac{1}{\varepsilon_r}(\nabla\times\vec{J}_i)_z=0 \quad (4)$$

In time-domain, (4) can be written as:

$$\frac{1}{\varepsilon_r}\nabla^2 H_z - \frac{\mu_r}{c_0^2}\left[\frac{\partial^2 H_z}{\partial t^2} + \frac{2\sigma_x}{\varepsilon_0}\frac{\partial H_z}{\partial t} + \frac{\sigma_x^2}{\varepsilon_0^2}H_z\right] = -\frac{1}{\varepsilon_r}(\nabla\times\vec{J}_i)_z \quad (5)$$

Let ω_c be the carrier frequency, and $V(t)$ and $j(t)$ be the envelope of the magnetic field and excitation current density, respectively. Then we can write H_z and $\vec{J}_i$ in a modulated signal format as follows:

$$H_z(t) = V(t)e^{j\omega_c t}$$
$$\vec{J}_i(t) = \vec{j}(t)e^{j\omega_c t} \quad (6)$$

Substituting (5) into (6), an envelope PDE is created:

$$\frac{1}{\varepsilon_r}\nabla^2 V(t) - \frac{\mu_r}{c_0^2}\left[(j\omega_c + \frac{\sigma_x}{\varepsilon_0})^2 + 2(j\omega_c + \frac{\sigma_x}{\varepsilon_0})\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right] \cdot V(t) = -\frac{1}{\varepsilon_r}(\nabla\times\vec{j}_i)_z \quad (7)$$

Testing equation (7) with a testing function T, the equation (7) is decomposed into:

$$\iint_s \left[\begin{array}{l}\frac{1}{\varepsilon_r}\nabla T\cdot\nabla V + \frac{\mu_r}{c_0^2}((j\omega_c + \frac{\sigma_x}{\varepsilon_0})^2 \\ +2(j\omega_c + \frac{\sigma_x}{\varepsilon_0})\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2})\cdot V\cdot T\end{array}\right] \cdot ds = \oint_\Gamma \frac{1}{\varepsilon_r}T\frac{\partial V}{\partial n}\cdot dl + \iint \frac{1}{\varepsilon_r}(\nabla\times\vec{j}_i)_z\cdot T\cdot ds \quad (8)$$

Consider a 2-D parallel waveguide example, PML is backed by Perfect Electric Conductor (PEC). Obviously, the path integral of equation (8) can be vanished. Using Galerkin method to solve (8), we get a difference equation:

$$[T]\frac{d^2 v}{dt^2} + [B]\frac{dv}{dt} + [S]v + f = 0 \quad (9)$$

$$T_{ij} = \iint_s \frac{\mu_r}{c_0^2}W_i W_j ds$$
$$B_{ij} = \iint_s \frac{2\mu_r}{c_0^2}(j\omega_c + \frac{\sigma_x}{\varepsilon_0})W_i W_j ds$$
$$S_{ij} = \iint_s \frac{1}{\varepsilon_r}\nabla W_i\cdot\nabla W_j + \frac{\mu_r}{c_0^2}(j\omega_c + \frac{\sigma_x}{\varepsilon_0})^2 W_i W_j ds$$
$$f_i = \iint_s -\frac{1}{\varepsilon_r}(\nabla\times\vec{j}_i)\cdot W_i ds \quad (10)$$

T, B, S are matrixes and f is a vector, they are defined by (10). Using Newmark-Beta formulation [1] to discretize (9) in time domain:

$$\left[\frac{[T]}{\Delta t^2} + \frac{[B]}{2\Delta t} + \frac{[S]}{4}\right]v(n+1) = \left[\frac{2[T]}{\Delta t^2} - \frac{[S]}{2}\right]v(n) + \left[-\frac{[T]}{\Delta t^2} + \frac{[B]}{2\Delta t} - \frac{[S]}{4}\right]v(n-1) - \frac{f(n+1) + 2f(n) + f(n-1)}{4} \quad (11)$$

To evaluate the performance of the PML, we set PML at the two ends of the parallel waveguide and excitation current density in the middle of computational area. In order to reduce the discretization error, we use spatially variant conductivity along the normal axis:

$$\sigma_x(x) = \frac{\sigma_{\max}|x - x_0|^m}{\sqrt{\varepsilon_r}d^m} \quad (12)$$

where x_0 is the interface of the PML region and non-PML region, d is the depth of the PML and m is the order the polynomial variation. Usually we choose m =2.

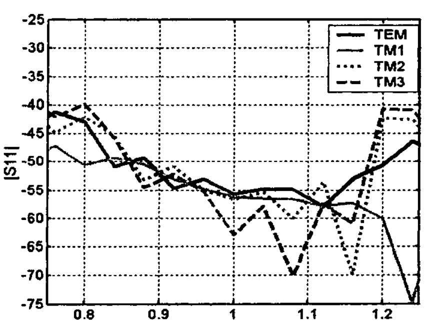

Fig. 1. Performance of PML for different TM modes

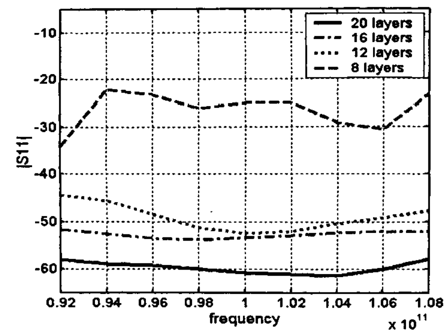

Fig. 2. Performance of PML for different layers

Fig. 1 shows the magnitude of S_{11} inside the empty waveguide for different TM modes. In this case, we choose $\Delta x = 0.0003m$, $\Delta y = 0.005m$, carrier frequency as 100GHz and the excitation Guassian pulse's -25db bandwidth is about 50GHz. The PML performance is tested by TEM, TM1, TM2, TM3 modes which are corresponding to different incident angels: $0°, 17.5°, 36.9°, 64.2°$. For different TM modes, the reflection of PML can be as small as -50db by carefully choosing σ_{max}.

Fig. 2 is the magnitude of S_{11} versus the different PML depths. All the parameters are the same as the previous case except that the excitation's bandwidth is about 16GHz. The result indicates that the larger the PML depth, the better the PML performs. This is because as we increase the depth, we must decrease the σ_{max}, thus the field inside the PML will decay more gradually so that the discretization error gets smaller. When we set the PML layers more than 16, we can easily get a reflection lower than -50db.

III ANALYSIS OF DISPERSIVE MEDIUM USING EVFE

The general frequency domain second order wave equation in dispersive medium can be written as:

$$\nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} = \omega^2 \mu_0 \varepsilon(\omega) \vec{E} - \mu_0 j\omega \vec{J}_s \tag{13}$$

First, consider a simple kind of dispersive medium plasma, therefore the complex permittivity is:

$$\varepsilon(\omega) = \varepsilon_0 + \frac{\omega_p^2 \varepsilon_0}{j\omega(j\omega + v_c)} \tag{14}$$

where ω_p denotes the plasma frequency, and v_c is the damping frequency. For simplicity, we assume that only the TEM mode exists in the plasma and the plasma is source free. Thus, equations (13) and (14) can be written in time domain forms as:

$$\frac{1}{\mu_r} \nabla^2 E_y + \frac{1}{c_0^2} \frac{\partial^2 E_y}{\partial t^2} - \frac{\omega_p^2}{c_0^2} \frac{\partial \varphi(t)}{\partial t} = 0 \tag{15}$$

$$\varphi(t) = \exp(-v_c t) \bar{u}(t) \otimes E_y(t) \tag{16}$$

where $\bar{u}(t)$ is the unit step function and $\otimes$ denotes the convolution.

Customarily, to solve equation (16), the "Recursive Convolution Method" [6] is used. Nevertheless, if this method is directly employed in EVFE technique, in most cases we cannot obtain an accurate or convergent result, because the time step in EVFE is usually so sparse that the integral's precision deteriorates quickly. Here "Mutual Difference Method" is employed, by which a highly precise result can be gained. We use the difference form of equation (16):

$$\frac{\partial \varphi(t)}{\partial t} + v_c \varphi(t) = E_y(t) \tag{17}$$

Then define the field and source in modulated signal format:

$$E_y(t) = U(t) e^{j\omega_c t}$$
$$\varphi(t) = \phi(t) e^{j\omega_c t} \tag{18}$$

Associate (16),(18),(19) and impose the EVFE solver. Ultimately two partial difference equations about the signal envelopes can be obtained:

$$[A]\frac{d^2 u}{dt^2} + [B]\frac{du}{dt} + [C]u + [D]\frac{d^2 \phi}{dt^2} + [E]\frac{d\phi}{dt} + [F]\phi = 0 \tag{19}$$

$$\frac{\partial \phi(t)}{\partial t} + (j\omega_c + v_c)\phi(t) = u(t) \tag{20}$$

Use Newmark-Beta formulation to discretize the signal envelopes $U(t)$ and $\phi(t)$ in equations (19) and (20). In time domain:

$$[K_1]u(n+1) = [K_2]u(n) + [K_3]u(n-1) + [K_4]\phi(n+1) + [K_5]\phi(n) + [K_6]\phi(n-1)$$
$$\phi(n+1) = a \cdot \phi(n) + b \cdot \phi(n-1) + c \cdot u(n+1) + d \cdot u(n) + e \cdot u(n-1) \tag{21}$$

$[K_i], i = 1,...,5$ are the system matrixes and a, b, c, d, e are constant numbers. After associating these two equations, and solving them, the complex signal envelope vectors in time domain will be obtained.

To testify the validity of the formulations derived with EVFE technique, we calculate the S11 and S21 of a plasma slab with a thickness of 2cm. The plasma frequency is $\omega_p = 28.7G \cdot 2\pi$ rad/s and damping frequency is $v_c = 20G$ rad/s.

Fig. 3 shows the computational model, which is a parallel waveguide partially filled with plasma. We use PML to truncate the EVFE meshes in order to temper the reflection error from the two ends. Since we are only interested about a small bandwidth (about 20Ghz) near the plasma frequency, thus we can set the carrier frequency for the modulated signals to be 31Ghz and let the bandwidth of the excitation Gaussian pulse be 20Ghz. Since the time step in EVFE technique is only limited by the signal's bandwidth, the time step is at least 4 times sparser than that is required by a FETD code.

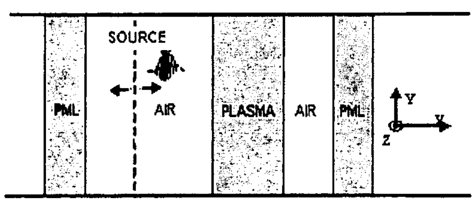

Fig. 3 The computational model for a plasma slab

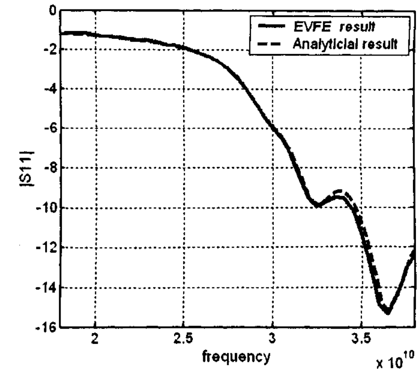

Fig. 4 Magnitude of S11 of the plasma slab

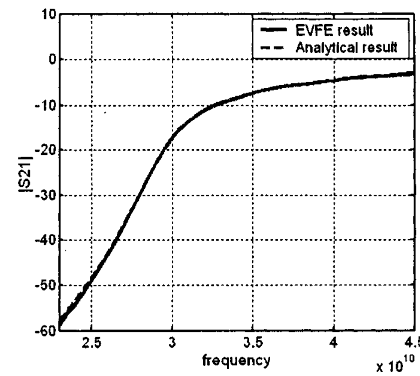

Fig. 5 Magnitude of S21 of the plasma slab

Fig. 4 and Fig. 5 is the magnitude of the plasma slab's S11 and S21, which denote the refection coefficient and transmission coefficient of the plasma slab. The result of EVFE with PML agrees well with the analytical solution using plane wave theory.

IV CONCLUSION

In this paper, the anisotropic PML has been implemented into the EVFE algorithm. Numerical examples have been presented to evaluate the PML's performance. More than 50db overall absorption is achieved when a 16 layers absorber is used. The EVFE formulations for dispersive medium are also derived here and a plasma example is presented to testify the validity of the formulations..

REFERENCES

[1] Yuanxun Wang and Tatsuo Itoh, "Envelope-finite element(EVFE) Technique – A more efficient time domain scheme", *IEEE Trans. Microwave Theory Tech.* Dec. 2001.

[2] S.D. Gedney, "An Anisotropic Perfectly Matched Layer Absorbing Medium for the Truncation of FDTD Lattices," *IEEE Trans. Antennas Propagat.*, vol.44, (no.12),pp.1630-1639, Dec. 1993.

[3] Hsiao-Ping Tsai, Yuanxun Wang and Tatsuo Itoh, "An Unconditionally Stable Extended (USE) Finite Element Time Domain Solution of Active Nonlinear Microwave Circuits Using Perfectly Matched Layers", accepted by *IEEE Trans. Microwave Theory Tech.*

[4] D. Jiao and J. M. Jin, "Time domain finite element modeling of dispersive media", *2001 IEEE AP-S,* Boston, Massachusetts, July 8-13, 2001.

[5] O. P. Gandhi, B. Q. Gao, J. Y. Chen, "A frequency-dependent finite-difference time domain formulation for general dispersive media," *IEEE Trans. Antennas Propagat.*, vol.41, (no.4),pp.658-665, April. 1993.

[6] D. H. Werner, R. Mittra, "Frontiers in Electromagnetics," *IEEE press*, 1999.